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Teaching Experiments in exploring convex and concave functions

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Abstract

Convex and concave functions are an important mathematical concept. Essential understanding of this concept will help gifted students construct applications in solving mathematical problems, including Olympiad mathematics. Using a constructivist teaching experiment, this study developed tasks that support students to explore and deepen their understanding of the concept of convex and concave functions. Experimentation using a visual model enabled students to form and verify hypotheses and construct their knowledge about convex and concave functions in an easier way.

Key words: Teaching experiment, Calculus, Secondary school mathematics, Dynamic Geometry Software, Lesson design research, Constructivism

Introduction

When presenting the concept of convex, concave functions, Vietnamese textbooks for Mathematical gifted students present the following definitions:
"Suppose *I* is an open interval of $\mathbb{R}$. A function f is called convex function on *I* if and only if for every $t \in [0;1]$ we have $f\left(tx+(1-t)x_0\right) \le tf\left(x\right)+(1-t)f\left(x_0\right)$.
In case $f\left(tx+(1-t)x_0\right) \ge tf\left(x\right)+(1-t)f\left(x_0\right)$.we say that f is a concave function on *I*.". (Quynh, Dung, Luong, & Thang, 2010, p. 237).
Then the textbooks present theorems and prove them. This presentation gives the students two difficult problems: First, students do not understand why convex and concave functions should satisfy these conditions. Secondly the second students do not understand the relationship between the terms "convex" and "concave" with the conditions in the definition. These difficulties prevent students to understand the significance of the condition and causer confusion and passivity.

In this paper, we focus on answering the following research questions
1 How can students construct knowledge about the convex and concave functions through experimenting with a visual model?
2 How can students be helped to use a visual model to explore knowledge about convex and concave functions?

Research framework and design

The role of intuition in the formation of knowledge
For concepts related to functions, the properties of a function can be represented on its graph. Thus, we can organize activities for students to discover new knowledge of a function from the graphs. A graph can give students visual images of the related key ideas and enable them to "see" the implications of these ideas.
According to Kant: "… all human cognition begins with observations, proceeds from thence to conceptions, and ends with ideas" (Polya, 1965, p. 103). In this sentence the

terms “observation”, “conception”, and “idea” were used. Polya re-expressed the sentence as: “Learning begins with action and perception, proceeds from thence to words and concepts, and should end with training certain new properties of intellectual gift” (p. 103).

When considering the role of logic, intuition and analysis, we can say that logic and analysis do not make mistakes, but they are limited. Intuition is a strong ally, after the powers of logic have been exhausted. Intuition is not a substitute for analysis, but it rather is a companion.

When talking about the role of intuition Courant and Robbins (1996) remarked that:

> There seems to be a great danger in the prevailing overemphasis on the deductive-postulational character of mathematics. True, the elemnet of constructive invention, of directing and motivating intuition, is apt to elude a simple philosophical formulation; but it remains the core of any mathematical achievement, even in the most abstract fields. If the crystallized deductive form is the goal, intuition and construction are at least the driving forces. (p. iii)

Thus, intuition is closely related to innovation and invention in the mathematical sciences. It is likely that many talented mathematicians and scientists, such as Newton and Einstein, operated with intuitive judgments involving mathematics or science before they used formal knowledge. Thus, starting from visualization to provide an opportunity for students to creatively propose ideas is essential. Therefore, in the design of the learning task, we ask students to rely on visual images to create new knowledge.

The depth of knowledge and understanding of the learning

Comprehension can be simply described as the students understand the knowledge meaningful to them. Comprehension occurs when students are able to connect old knowledge with new knowledge enabling new information to be connected with the old knowledge, and so facilitating the development of new creations.

However, teachers need to develop a plan for the issues, themes and ideas thoroughly and think about how the formation of new knowledge by linking it to old knowledge. Teaching to comprehend new ideas, in this sense, includes complex learning activities, such as thinking about how to approach a task to solve the problem. Teaching to promote deeper learning requires students to move from the surface of an overarching theme to find out more detailed knowledge on the topic. Clear teaching strategies are needed to help students recognize the relationships and connect concepts and ideas.

Research Design

In this research design, the following factors are key considerations: the teacher design schedule, the teaching design idea, the design of mathematical tasks, data collection and analysis.

The idea of teaching design

The concept of a convex and concave functions is difficult to teach and understand. However, the words "convex" and "concave" in the definition are intuitive. The idea of teaching design is: to first, create activities combined with visual images through which

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students develop self-defined conditions. This helps students to clarify the existence of each of the defined conditions. Then, it is necessary to create activities for students to discover new knowledge from this definition. To help students access the concepts and facilitate the creation of knowledge, in the teaching process, four perspectives – numerical, graphical, language and algebraic – were utilized to address this challenge. We think it is important for learning that students perceive the process of concept formation and through that process form knowledge about method and develop new ideas. Therefore, in order to help students demonstrate their understanding of the learning process, after students have formed the concept, we set up open-ended questions to provide an opportunity to allow students to propose innovative ideas. The innovative ideas proposed were conducted based on two directions: Based on the manipulation of thinking such as similarity, specific conditions, or generalize based on visual images.

A teaching design schedule

In designing our teaching experiment, preparing teaching and learning activities, we adopted a Constructivist Learning approach following Confrey (1991) and Stephens (2012). The following four points indicate how we have combined the two key ideas of a teaching experiment within a constructivist-learning framework:

- Our teaching experiment needs to focus on the important key conceptions of the convex and concave functions.
- The teacher should have a clear plan to respond students' incorrect answers.
- The teacher should have a longer-term plan to consistently develop students' deep understanding of the convex and concave functions.
- The teaching experiment should give concrete examples that are familiar and easy for students to help them understand the convex and concave functions.

Design of mathematical tasks

To obtain data to answer the key research questions, we designed activities and took students in a Grade 11 specialized mathematics through a range of learning experiments. In the design of the questions, we launched open-ended questions to provide students with the opportunity to express their thoughts on their experience and to raise ideas that may have been formed while experiencing the mathematical model.

Task 1

Let graph of function $y = f(x)$ (Figure 1). Manipulate with the model and compare the value of *y*-coordinate of the point on graph with the point on a straight line passing through two points $M_0(x_0; f(x_0)), M(x; f(x))$ that they have the same *x*-coordinate. [M and N need to be shown on the graph. Please explain the blue lines. I know what they are, but it should be stated clearly.]

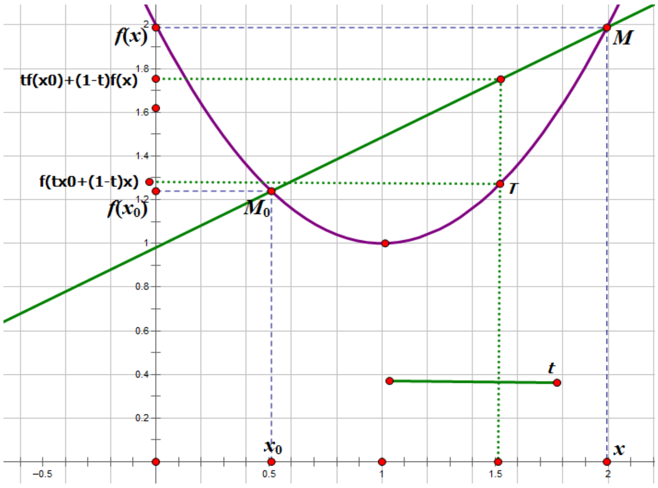

Figure 1. Graph of a function $f(x)$ convex on I

Task 2

Consider the graph convex function $y = f(x)$ on the interval I contain x_0, x and $t \in (0;1]$, as shown in Figure 2

a. From the definition of convex and concave functions specific with conditions, generalize to propose new results.

b. Write the tangent equation of the graph of function at the point that *x*-coordinate of it as *a*. Compare the value of *y*-coordinate of the point on graph with the point on tangent that they have the same *x*-coordinate (Figure 2). From there, to propose new related results

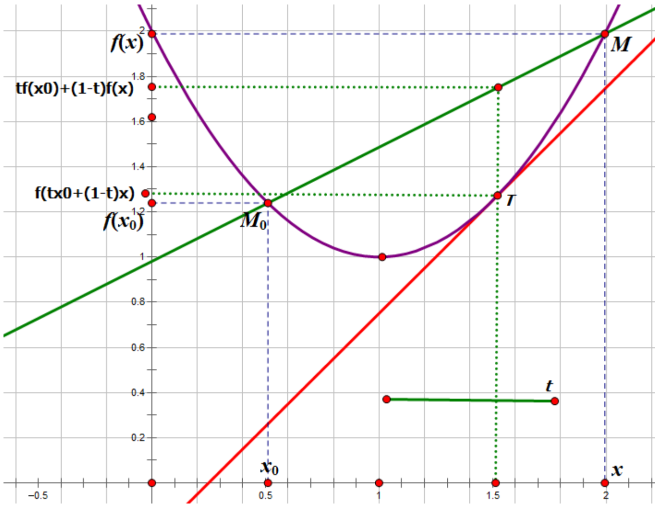

Figure 2. Graph of a function $f(x)$ convex on I and its tangent

Research Results

In this section we focus on the following results: The interaction between the students and teachers to support students facing difficulty, processing by the teacher of correct results, and teachers responding to incorrect results

Collaboration among students

In the process of monitoring the work between the groups, the majority of students actively worked to produce results for the groups. All groups worked with a visual model, and members observed and discussed with each other to answer questions in the tasks.

Teacher support for students facing difficulty

In the process of monitoring the implementation of group activities “if students have difficulties answering these questions, the teacher poses additional questions or make additional requests to support students.” (Nam P. S., Stephens, M (2014))

When performing the study respondents task 1, there was a group wondering why the number $x_0+t(x-x_0), t\in[0;1]$ belongs to $[x_0;x]$. To help students answer these concerns, we asked the question "What comments do you have about the value $x-x_0$?", "when t changes on $[0;1]$, what comments do you have on the function $g(t)=x_0+t(x-x_0)$ of the variable t and the its value on $[0;1]$". These questions were to help students realize that when $x\neq a$ then $g(t)=x_0+t(x-x_0)$ is increase on $[0;1]$ and thus its value belongs to $[g(0);\mathrm{g}(1)]$ or $[x_0;x]$.

There were several groups of students facing difficulties in answering task 1. In this case, teachers set requirements: "Take a point on the graph and a point on segment M_0M such that they have same horizontal. Then compare their *y*-coordinates"

For task 2, some groups were confused when they saw the words "specific conditions" and "generalize". In this case, the teacher explained students to understand these terms. Some groups initially only produced one or two results, teachers can suggest example like "If t take the special value $\frac{1}{2},\frac{1}{3}$ then what were will be the results".

For question 3, there were some groups that did not know how to compare the value of *y*-coordinate, in this case, we suggest: "Draw a line perpendicular to the horizontal axis at the point with *x*-coordinate, define the their intersection the graph of $y=f(x)$ and its tangent, then compare the values of the *y*-coordinate".

Processing by teacher of correct results

Many sound results were given by these groups. Students were asked to clarify where the results came from and prove them (if possible). Some results may still not be able to be proved, but these were valuable, because the discovery of new results motivated students in seeking a deeper understanding.

In the results obtained from task 1, all groups obtained the following result “$f(tx+(1-t)x_0)\leq tf(x)+(1-t)f(x_0)$.”. When asked to explain, there were three groups that gave the following response or its close equivalent: "the graph of function always lies below the straight line connecting two points, which means that at the same *x*-coordinate $x_0+t\left(x-a\right)=tx+(1-t)x_0$ then *y*-coordinate of the point on the graph is always less than or equal to *y*-coordinate of the point on the line M_0M ". However, the *y*-coordinate of the point on the graph is $f\left(tx+(1-t)x_0\right)$, and the *y*-coordinate of the point on the segment M_0M is $tf\left(x\right)+(1-t)f\left(x_0\right)$. So, we have inequality $f\left(tx+\left(1-t\right)x_0\right)\le tf\left(x\right)+\left(1-t\right)f\left(x_0\right)$." Then the teacher required students to observe the graph of the function to detect its "depressions". Since then, we have reviewed "concave function on *I*, then for all $t\in\left[0;1\right]$ we have. $f\left(tx+\left(1-t\right)x_0\right)\le tf\left(x\right)+\left(1-t\right)f\left(x_0\right)$.". From here, teachers provided a definition of a concave function. The teacher asked students to make another statement. From here, results such as the following were received: " f is convex on $\left[x_0;x\right]$ if and only if $y\in\left[x_0;x\right]$ $f\left(y\right)\le\lambda f\left(x\right)+\left(1-\lambda\right)f\left(a\right),\ t\in\left[0;1\right]$".

When asked further, "If f is a concave function on $\left[x_0;x\right]$, then what can you comment on the value $f\left(tx+\left(1-t\right)x_0\right);tf\left(x\right)+\left(1-t\right)f\left(x_0\right)$", all groups gave a similar answer. $f\left(tx+\left(1-t\right)x_0\right)\ge tf\left(x\right)+\left(1-t\right)f\left(x_0\right)$.

When performing task 2, sound results were also given.

Results 1 If the graph of function $f(x)$ is convex on an open interval *I* containing a,b then $f\left(\frac{a+b}{2}\right)\le\frac{1}{2}\left[f\left(a\right)+f\left(b\right)\right]$.

Result 2 If the graph of function $f(x)$ is concave on an open interval *I* containing a,b then $f\left(\frac{a+b}{2}\right)\ge\frac{1}{2}\left[f\left(a\right)+f\left(b\right)\right]$.

Result 3 If the graph of function $f(x)$ is convex on an open interval *I* containing a,b then

$$f\left(\frac{2a+b}{3}\right)\le\frac{1}{3}\left[2f\left(a\right)+f\left(b\right)\right].$$

Result 4 If the graph of function $f(x)$ is concave on an open interval *I* containing a,b then

$$f\left(\frac{2a+b}{3}\right)\ge\frac{1}{3}\left[2f\left(a\right)+f\left(b\right)\right].$$

By generalization, most students got the following results:

Result 5: If the graph of function $f(x)$ is convex on a open interval *I* containing $x_1,\ldots,x_n,a$

$$f(x_1)+f(x_2)+\ldots+f(x_n)\le f'(a)(x_1+\ldots+x_n-na)+nf(a),\forall x_1,\ldots,x_n\in I$$

Result 6 If the graph of function $f(x)$ is concave on a open interval *I* containing

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$x_1,...,x_n,a$ then

$f(x_1)+f(x_2)+...+f(x_n)\geq f'(a)(x_1+...+x_n-na)+nf(a),\forall x_1,...,x_n\in I$

For the question b of task 2, the majority of groups answered "If the graph of $f(x)$ is convex on interval $(c;b)$ contains a, x then. $f(x)\geq f'(a)(x-a)+f(a),\ \forall x\in(c;b)$" When asked to explain this, two groups argued that "tangent at the point that x-coordinate of it as a always lies below the graph, which means that at the same x-coordinate, the y-coordinate of the tangent point is always less than or equal to y-coordinate of the point on the graph. However, y-coordinate of the point on the graph is $f(x)$, and the y-coordinate of the point on the tangent is $f'(a)(x-a)+f(a)$ or we have the inequality $f(x)\geq f'(a)(x-a)+f(a)$". Then students were asked an additional question: "Please explain the meaning of the findings." At the same time, questions also provided opportunities for students to recognize a new idea in the creation of inequality, if it is convex or concave graph that on $(a;b)$, any point $(a;b)$ by taking and use of the property equations, we will get the new inequality.

Then students were asked an additional question: "consider the function $f(x)=6x^3-x^2$ and propose new related results". Sound result was given: "the function $f(x)=6x^3-x^2$ is convex on $(0;1)$ and the equation of the tangent line at $x=\frac{1}{4}$ is $y=\frac{1}{8}(5x-1)$. So we claim that for $0<x<1$, $6x^3-x^2\geq\frac{1}{8}(5x-1)$ or $48x^3-8x^2-5x+1\geq 0$." We also encouraged students chosen other specific function and obtained results: "choose $f(x)=\frac{x^2+2x+1}{3x^2-2x+1}, a=\frac{1}{3}$ we have $\frac{x^2+2x+1}{3x^2-2x+1}\leq\frac{12x+4}{3}$ for $0<x<1$.", "$\frac{1}{x^2+1}\leq\frac{27}{50}(-x+2)$ for $0<x<1$".

Also from the inequality $f(x)\geq f'(a)(x-a)+f(a)$, by choosing a specific function and a specific value, we will obtain various inequalities. These results allow us to create inequality, as well as give us a way to prove the inequality that we can tentatively called "the tangent method".

Teachers responding to incorrect results

In discussing students' results, there were several groups that gave incorrect or incomplete answers. To help students recognize these mistakes, we proceeded as follows: If the misconceptions in the class and ask all of the control group. Teachers can use the following methods to support verification of the student, which are proposed by Nam P. S., Stephens, M (2013):

• Teachers provide a counter-example and ask students to examine and compare the answers

• Teachers ask students to undertake further activities through activities that allow students to realize the mistake.

• Teachers ask students to use their knowledge to test their results, because the results they give were often based on images obtained on a model that had not been examined closely.

For task 1, there were two groups that said " $f\left(tx+(1-t)x_0\right)<tf\left(x\right)+(1-t)f\left(x_0\right)$.". To help students realize this mistake, the teacher posed the question "Compare the values of *y*-coordinate in the cases $t=0, t=1$". As a result, students realized that when $t=0, t=1$ $f\left(tx+(1-t)x_0\right)=tf\left(x\right)+(1-t)f\left(x_0\right)$.

When groups answered the question *b* of task 2, some students made similar mistakes in assuming that "If the convex graph $y=f\left(x\right)$ on $(c;b)$ contains *a*, *x* then $f\left(tx+(1-t)x_0\right)>tf\left(x\right)+(1-t)f\left(x_0\right)$. $\forall x\in(c;b)$". To help students recognize this mistake, the teacher could ask the question: "Find the value of $f(x), f'(a)(x-a)+f(a)$ at $x=a$". Having ascertained that the resulting values of the two expressions were equal helped students to recognize mistakes and to adjust their results.

Discussion and Conclusion

Convex and concave functions are difficult concepts in calculus. However, there is an immediate advantage in building on those elements of the concept that are "intuitive". Using contexts derived from visual models created favorable conditions for students to construct new knowledge. Designing a task-based learning approach with open-ended questions and with specific guidance from the teacher created opportunities for students to connect new ideas to what they already knew and to construct deeper knowledge.

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